Supplement: Complex Numbers

- The complex numbers consist of numbers of the form $z = x + iy$ where $x, y \in \mathbb{R}$ and $i^2 = -1$
  - the real part of z is x: $\mathrm{Re}(z) = x$
  - the imaginary part is y: $\mathrm{Im}(z) = y$

- We define the complex conjugate of $z = x + iy$
  $\bar{z} = x - iy$

  So $\mathrm{Re}(z) = \frac{z + \bar{z}}{2}$, $\mathrm{Im}(z) = \frac{z - \bar{z}}{2i}$

- Note $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ and $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

- If $z \neq 0$, $\frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}$ is its inverse

  e.g.) $1/i = -i/(i)(-i) = -i$

- We define the magnitude of z to be $|z| = \sqrt{x^2 + y^2}$
  or $|z| = \sqrt{z\bar{z}}$

- The theory of sequences & series carries over with minor changes: $\lim_{k \to \infty} z_k = z$ if $\lim_{k \to \infty} |z - z_k| = 0$

- A series $\sum a_k$ converges if the sequence $\{\sum_{k=1}^{n} a_k\}_{n=1}^{\infty}$ does, and converges absolutely if
  $\sum_{k=1}^{\infty} |a_k| < \infty$.

• One famous absolutely convergent series is the power series

$$e^z = \sum_{k=0}^{\infty} z^k/k!$$

which converges for all z. By splitting real & imaginary parts,

$$e^{i\theta} = \left(1 - \theta^2/2! + \theta^4/4! + \dots\right) + i\left(\theta/1! - \theta^3/3! + \theta^5/5!\right)$$

$$= \sum_{n=0}^{\infty} \frac{\theta^{2n}}{(2n)!}(-1)^n + i \sum_{n=0}^{\infty} \frac{\theta^{2n+1}}{(2n+1)!}(-1)^n$$

$$= \cos(\theta) + i\sin(\theta)$$

This yields natural polar representation

$$z = re^{i\theta}$$

• We can use this power series to derive common properties

$$e^{z+w} = e^z e^w$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha+\beta) = \cancel{\sin(\alpha)\sin(\beta)}\ \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)$$

• ex.) $$\frac{d}{dx} e^{ax} = \frac{\partial}{\partial x}\sum_{k=0}^{\infty} \frac{a^k x^k}{k!} \underset{\text{By absolute convergence}}{=} \sum_{k=0}^{\infty} \frac{a^k \frac{d}{dx} x^k}{k!} = \sum_{k=0}^{\infty} \frac{a^k k x^{k-1}}{k!}$$

$$= \sum_{k=1}^{\infty} a \cdot \frac{a^{k-1} x^{k-1}}{(k-1)!} = ae^{ax}$$